Last time: let $B \in \mathbb{R}^{n \times n}$

- $\mathrm{Spec}(B)$ = multiset of eigenvalues of B, counted with **algebraic multiplicity**

$$= \{ \underbrace{\lambda_1, \dots, \lambda_1}_{d_1 \text{ times}}, \dots\dots, \underbrace{\lambda_r, \dots, \lambda_r}_{d_r \text{ times}} \}$$

$$\chi_B(t) = \det(t \cdot I_n - B) = (t-\lambda_1)^{d_1} \cdots (t-\lambda_r)^{d_r}$$

$\det(-X) = (-1)^n \det(X)$

- **geometric multiplicity** of an eigenvalue λ of B is

$$\dim \mathrm{Ker}(B - \lambda \cdot I_n) \leq \text{algebraic multiplicity}$$

- similar matrices ($A \sim B$ if $B = PAP^{-1}$) have the same Spec

- for a real matrix, λ eigenvalue $\Longleftrightarrow$ $\bar{\lambda}$ eigenvalue

$$\begin{array}{ccc} \lambda \text{ eigenvalue} & \Longleftrightarrow & \bar{\lambda} \text{ eigenvalue} \\ \updownarrow & & \updownarrow \\ v \text{ eigenvector} & \Longleftrightarrow & \bar{v} \text{ eigenvector} \end{array}$$

(moreover, λ and $\bar{\lambda}$ have the same algebraic/geometric multiplicities)

Note: $\mathbb{R} \subset \mathbb{C}$, $a = a + 0 \cdot i$; if a is real, $\bar{a} = a$

$\updownarrow$

v eigenvector, $\bar{v} = v$

Today: for $B = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$, λ has algebraic multiplicity n $=$ geometric multiplicity n

Let for $B = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \end{pmatrix}$, λ has algebraic multiplicity n $\geq$

but for $B = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$, λ has geometric multiplicity 1

- B is triangular, $\mathrm{Spec}(B) = \{\underbrace{\lambda, \dots, \lambda}_{n \text{ times}}\}$

- $\dim \mathrm{Ker}(B - \lambda \cdot I_n) = \dim \mathrm{Ker} \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & 0 & 1 \\ 0 & & & 0 \end{pmatrix}$

$$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ is in } \underbrace{\mathrm{Ker} \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}}_{\mathrm{Span}\, e_1} \iff 0 = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_2 \\ \vdots \\ x_n \\ 0 \end{pmatrix}$$

$$\iff x_2 = \dots = x_n = 0$$

DEF 20.1: the **eigenspace** of B w.r.t the eigenvalue λ is $V_\lambda = \mathrm{Ker}(B - \lambda \cdot I_n)$

$= \{0\} \cup \{\text{eigenvectors for } \lambda\}$

$$v \in V_\lambda = \mathrm{Ker}(B - \lambda I_n) \iff (B - \lambda I_n) v = 0$$

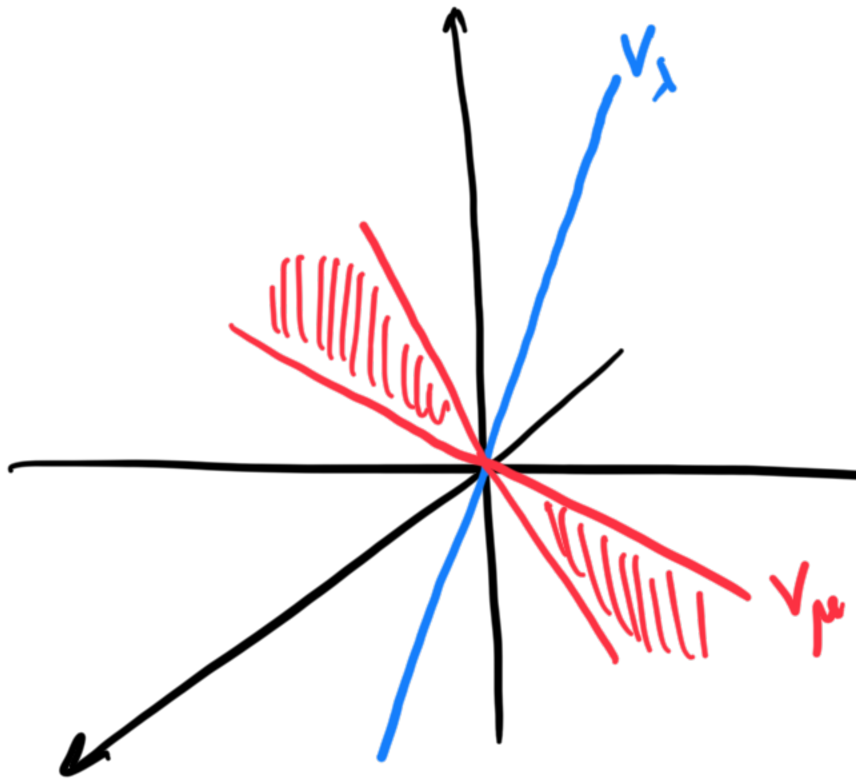
$$\iff Bv = \lambda v$$

$\iff$ "v is an eigenvector for λ" or "$v = 0$"

($\underbrace{\text{geometric multiplicity of } \lambda \text{ is then } \dim V_\lambda}_{\text{maximal number of linearly independent eigenvectors for } \lambda}$)

Note: V_λ is a subspace of $\mathbb{R}^n$, $\forall$ eigenvalue λ

Example in $\mathbb{R}^3$



$B \in \mathbb{R}^{3\times 3}$

λ has geom. mult 1

μ has geom mult 2

because $1+2=3$ the geom. mult are as big as possible

$\Downarrow$

B is diagonalizable

THM 20.2: the different eigenspaces of any matrix B are linearly independent, i.e. anyhow you choose one eigenvector in each eigenspace, these eigenvectors will be independent

suppose $\lambda_1, \dots, \lambda_r$ are the distinct eigenvalues of B

$V_{\lambda_1} \quad V_{\lambda_r}$ are subspaces of $\mathbb{R}^n$

$$\left.\begin{array}{l} \forall\, 0 \neq v_1 \in V_{\lambda_1} \\ \quad\vdots \\ 0 \neq v_r \in V_{\lambda_r} \end{array}\right\}, \ \{v_1, \ldots, v_r\} \text{ will be linearly independent}$$

COR 20.3: any two eigenspaces intersect only in 0

$$\Updownarrow$$

v cannot simultaneously be an eigenvector for $\lambda \neq \mu$

(Proof: $\lambda v = Bv = \mu v \Rightarrow (\lambda - \mu) v = 0 \Rightarrow v = 0$)

Proof of Thm 20.2: suppose $0 \neq v_1 \in V_{\lambda_1}, \ldots, 0 \neq v_r \in V_{\lambda_r}$ are linearly dependent $\Rightarrow v_r = \alpha_1 v_1 + \ldots + \alpha_{r-1} v_{r-1}$

$$\Downarrow$$

$$B v_r = \alpha_1 B v_1 + \ldots + \alpha_{r-1} B v_{r-1}$$

$$\Downarrow$$

$$\lambda_r v_r = \alpha_1 \lambda_1 v_1 + \ldots + \alpha_{r-1} \lambda_{r-1} v_{r-1}$$

$\cdot \lambda_r$

$$\lambda_r v_r = \alpha_1 \lambda_r v_1 + \ldots + \alpha_{r-1} \lambda_r v_{r-1}$$

$(-)$

$v_1, \ldots, v_{r-1}$ are lin. dependent $\Leftarrow$ $0 = (\lambda_1 - \lambda_r)\alpha_1 v_1 + \ldots + (\lambda_{r-1} - \lambda_r)\alpha_{r-1} v_{r-1}$

$$\Downarrow$$

$v_1, \ldots, v_{r-2}$ are dependent

$$\Downarrow$$

$$\vdots$$

$$\Downarrow$$

$0 \neq v_1$ is linearly dependent, impossible $\Rightarrow$ contradiction.

- $\mathrm{Spec}(B) = \{\lambda_1, \dots, \lambda_n\}$ then $\boxed{\lambda_1 \cdots \lambda_n = \det(B)}$

Proof: $\det(t \cdot I_n - B) = \chi_B(t) = (t-\lambda_1) \cdots (t-\lambda_n)$

set $t=0$: $\det(-B) = (-\lambda_1) \cdots (-\lambda_n)$

$$\cancel{(-1)^n} \det(B) = \cancel{(-1)^n} \lambda_1 \cdots \lambda_n$$

DEF 20.4: $B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \in \mathbb{R}^{n \times n}$

the **trace** of B is $\mathrm{tr}(B) = b_{11} + b_{22} + \dots + b_{nn}$

THM 20.5: $\forall\, B \in \mathbb{R}^{n \times n}$, $\boxed{\lambda_1 + \dots + \lambda_n = \mathrm{tr}(B)}$

(not necessarily diagonalizable)

PROP 20.6: let $X \in \mathbb{R}^{m \times n}$, $Y \in \mathbb{R}^{n \times m}$

$$\mathrm{tr}(\underbrace{XY}_{m \times m}) = \mathrm{tr}(\underbrace{YX}_{n \times n})$$

Proof: $X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \end{pmatrix}$ $Y = \begin{pmatrix} y_{11} & \cdots & y_{1m} \end{pmatrix}$

Proof: $X = \begin{pmatrix} \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}$ $\quad Y = \begin{pmatrix} \vdots & & \vdots \\ y_{n1} & \cdots & y_{nm} \end{pmatrix}$

$$XY = \begin{pmatrix} x_{11}y_{11} + x_{12}y_{21} + \dots + x_{1n}y_{n1} & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & x_{m1}y_{1m} + x_{m2}y_{2m} + \dots + x_{mn}y_{nm} \end{pmatrix}$$

$$YX = \begin{pmatrix} y_{11}x_{11} + y_{12}x_{21} + \dots + y_{1m}x_{m1} & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & y_{n1}x_{1n} + y_{n2}x_{2n} + \dots + y_{nm}x_{mn} \end{pmatrix}$$

$$\mathrm{tr}(XY) = \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} x_{ij} y_{ji} = \mathrm{tr}(YX) \qquad \square$$

Proof of Thm 20.5 for diagonalizable B

$$B = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^{-1}$$

$$(v_1 | \cdots | v_n) \quad \text{where } v_1 \in V_{\lambda_1}, \ldots, v_n \in V_{\lambda_n}$$

$$X := P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad Y := P^{-1}$$

$\xRightarrow{\text{Prop } 20.6}$

$$\operatorname{tr}(B) = \operatorname{tr}(XY) = \operatorname{tr}(YX) = \operatorname{tr}\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = \lambda_1 + \cdots + \lambda_n$$

$\square$

$$B = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ 5 & 0 & 4 \end{pmatrix}$$

eigenvalues
eigenvectors
diagonalizability

$$\chi_B(t) = \det(t \cdot I_3 - B) = \det \begin{pmatrix} t-2 & 0 & 3 \\ 0 & t-1 & 0 \\ -5 & 0 & t-4 \end{pmatrix}$$

Cofactor expansion along row 2

$$= (-1)^{2+1} 0 + (-1)^{2+2} (t-1) \det \begin{pmatrix} t-2 & 3 \\ -5 & t-4 \end{pmatrix} + (-1)^{2+3} 0$$

$$= (t-1)\Big((t-2)(t-4) + 15 \Big)$$

$$= (t-1)(t^2 - 6t + 23)$$

$$= t^3 - 7t^2 + 29t - 23$$

Guess a root! $t=-1$ is not a root

$t=0$ is not a root

$t=1$ is a root, b/c $1-7+29-23=0$

$\exists\, a,b,c$ s.t. $t^3-7t^2+29t-23 = (t-1)(at^2+bt+c)$

$$t^3-7t^2+29t-23 = at^3 + bt^2 + ct$$
$$-at^2 - bt - c$$

$a=1$ $\quad$ $b-a=-7$ $\quad$ $c-b=29$ $\quad$ -23

$b=-6$ $\quad$ $c=23$

$$t^3-7t^2+29t-23 = (t-1)(t^2-6t+23)$$

roots

$$\lambda_1 = \frac{6+\sqrt{36-92}}{2} = \frac{6+\sqrt{-56}}{2} = 3 + i\cdot\sqrt{14}$$

$$\lambda_2 = \ldots\ldots\ldots\ldots = 3 - i\sqrt{14}$$

eigenvalues

$$t^3 - 7t^2 + 29t - 23 = (t-1)^1 \left(t-3-i\sqrt{14}\right)^1 \left(t-3+i\sqrt{14}\right)^1$$

$t^3 - 7t^2 + 29t - 23 = (t-1)(t - 3 \ldots)(\ldots)$

$\lambda_3 = 1$; all alg. mult. are 1

$$V_{\lambda_1} = \mathrm{Ker}(B - \lambda_1 I_3) = \mathrm{Ker}\begin{pmatrix} -1-i\sqrt{14} & 0 & -3 \\ 0 & -2-i\sqrt{14} & 0 \\ 5 & 0 & 1-i\sqrt{14} \end{pmatrix}$$

$\lambda_1 = 3 + i\sqrt{14}$

$$V_{\lambda_2} = \mathrm{Ker}(B - \lambda_2 I_3) = \mathrm{Ker}\begin{pmatrix} -1+i\sqrt{14} & 0 & -3 \\ 0 & -2+i\sqrt{14} & 0 \\ 5 & 0 & 1+i\sqrt{14} \end{pmatrix}$$

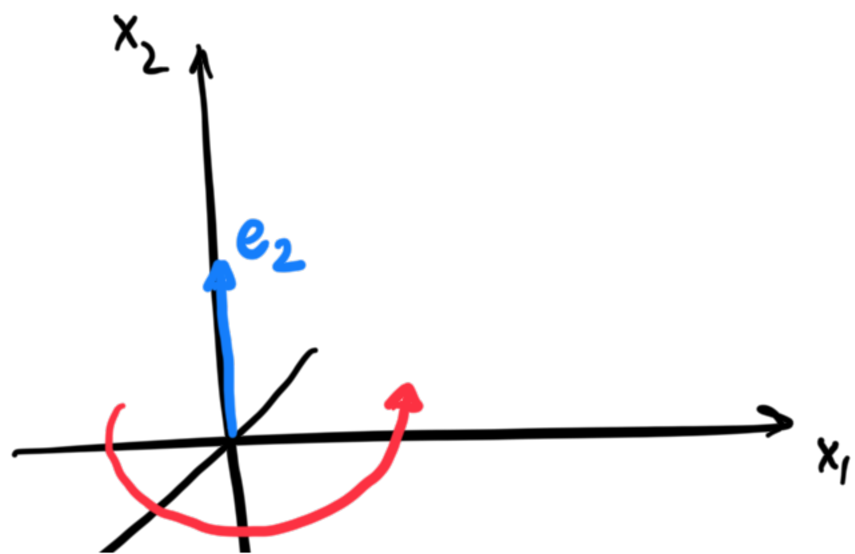
$\lambda_2 = 3 - i\sqrt{14}$

$$V_3 = \mathrm{Ker}(B - \lambda_3 I) = \mathrm{Ker}\begin{pmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 5 & 0 & 3 \end{pmatrix}$$

$\lambda_3 = 1$

$$= \mathrm{span}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e_2$$

$B : \mathbb{R}^3 \to \mathbb{R}^3$



Trick for calculating block determinants

(not an official part of the course, but could be helpful to you lateron)

$$B = \begin{pmatrix} 7\times 7 & * & * & * \\ 0 & 2\times 2 & * & * \\ 0 & 0 & 5\times 5 & * \\ 0 & 0 & 0 & 3\times 3 \end{pmatrix}$$

Then $\det(B) = \det(7\times 7)\,\det(2\times 2)\,\det(5\times 5)\,\det(3\times 3)$

(also true if B is lower triangular)

Example: for $\begin{pmatrix} t-2 & 0 & 3 \\ 0 & t-1 & 0 \\ -5 & 0 & t-4 \end{pmatrix}$, you can swap last

each swap multiplies det by -1

two rows and last two columns to get

$$\begin{pmatrix} t-2 & 3 & 0 \\ -5 & t-4 & 0 \\ 0 & 0 & t-1 \end{pmatrix}, \text{ whose det is } \det\begin{pmatrix} t-2 & 3 \\ -5 & t-4 \end{pmatrix} \cdot (t-1)$$

Still, for a general matrix the only ways to compute determinants are Gaussian elimination and cofactor expansion